The Surprising Versatility of Edge-Matching Tiles
as shown by their latest embodiment, MiniMatch-I

Kate Jones
President, Kadon Enterprises, Inc.
1227 Lorene Dr., Pasadena, MD 21122 USA
kate@gamepuzzles.com

Abstract

I present the results of an exploration of color distribution and topological equivalences with a set of 9 unique edge-colored square tiles using 4 colors, harkening back to Major Percy MacMahon's Three-Color Squares as described in his *New Mathematical Pastimes*, 1921. The delight of such a puzzle is the artistic surprise that each new solution brings. You could call it "math as moveable art".

Introduction

The subject of this article is a puzzle of just 9 pieces that can form a 3x3 square in 47,563,407,360 different ways, not counting rotations and reflections. It is the latest embodiment of a concept that dates back 95 years.

The idea of color-coding the four edges of a square to make them all different yet matchable to their neighbors goes back to a British mathematician, Major Percy MacMahon, and his 1921 book, *New Mathematical Pastimes* [1]. He proposed using three colors to mark the edges of squares, yielding 24 unique tiles to form a 4x6 rectangle [Figure 1, left] with joined sides matching and a uniform border color.

A few decades later, in the 1970s, an engineer from Lima, Ohio, by name of Wade Philpott, pioneered computer search programs on a now antique TRS-80 to identify that MacMahon's 3-Color Squares could have 20 different border configurations and 13,328 solutions. More about Wade's work is here: www.ucalgary.ca/lib-old/SpecColl/philpott.htm [2] and www.gamepuzzles.com/wade.htm [3].

Adding a fourth color raises the number of tiles to 70, fitting into a 7x10 rectangle, again with all matched edges and uniform border color—the Grand Multimatch I (Figure 1, center).

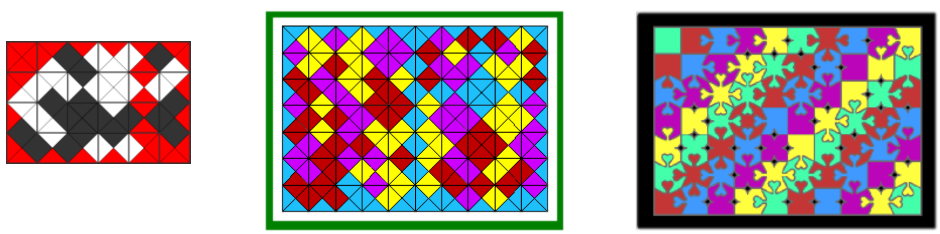


Figure 1: *MacMahon's Three-Color Squares (left), Grand Multimatch I (center) and Grand Snowflake (right).*

Fast forward to the Bridges Conference 2015 in Baltimore, MD. The Grand Snowflake, a contoured equivalent of Grand Multimatch I, was chosen for the 2015 art exhibit (Figure 1, right).

To give visitors an experience of how edge-matching works, I designed a miniature set (Figure 2) of only 9 pieces (a subset of Grand Multimatch-I), using 4 colors instead of contours of the Snowflake set. Leaving the little puzzle open next to the display gave visitors a fun activity with three simple challenges:

1. get all the tiles into the square with matched edges;
2. have one color entirely on the inside, with everything still matching;
3. have two opposite borders a single color, with everything still matching.

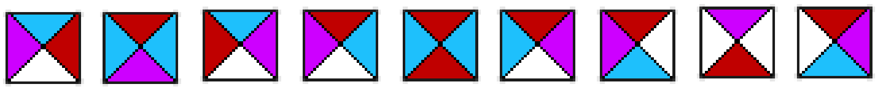


Figure 2: *The 9 tiles of MiniMatch-I.*

At the close of Bridges 2015, further explorations of the little puzzle, now named MiniMatch-I, found a surprising richness of variation. Highlights are shown below. All solutions were found by hand.

Definition of the MiniMatch-I set

The designing of MiniMatch-I presented some problems: the four colors filling 36 triangles (4 each on 9 squares) could not be divided evenly into four groups of 9, as an odd number could not make matching pairs. To match in any combination, each color had to occur on an even number of sides. Six of the tiles represented all the different ways the four colors could be placed on the square and made 3 mirror pairs. Three more tiles were needed, and rather than duplicate any of the six, I added "bowties", where opposite edges shared a color.

It took much experimentation to find which colors to put on the bowtie pieces to make the largest variety of goals solvable. Revisions and substitutions finally produced two bowtie pieces with three colors each and the third one with just two. One color does not have a bowtie at all. Thus two colors ended up with 10 triangles each, and the other two colors had 8. This division allowed those colors to form 5 and 4 internal squares, respectively, as well as make matches between opposite borders. Each color also had to be able to form solid opposite edges. More about this set is here: www.gamepuzzles.com/edgmtch5.htm#Mim1.

Research Results

Enclose any one color. Any one of the four colors can be totally enclosed with a fully edge-matched solution. Figure 3 shows, left to right: white, red, purple, blue. Bonus: pairs of matched borders.

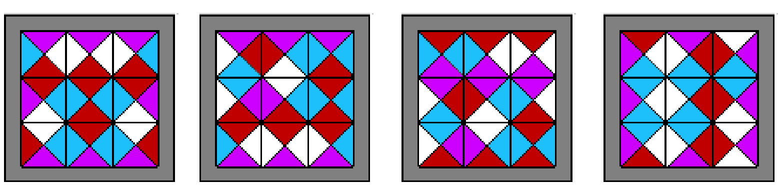


Figure 3: *One color fully enclosed.*

Enclose two colors. Certain two colors can be simultaneously enclosed (leaving just two colors on the border). Shown in Figure 4, purple/white and red/white. The other four color pairs—white/blue, purple/red, purple/blue and blue/red—are not solvable because of a shortage of tiles with the needed colors to complete the border and to match in the interior. White as a minority color is easier to enclose.

Purple, while also a minority, has no “bowtie” and cannot interface where two purple edges unavoidably face each other.

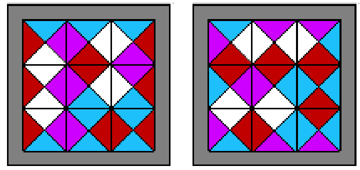
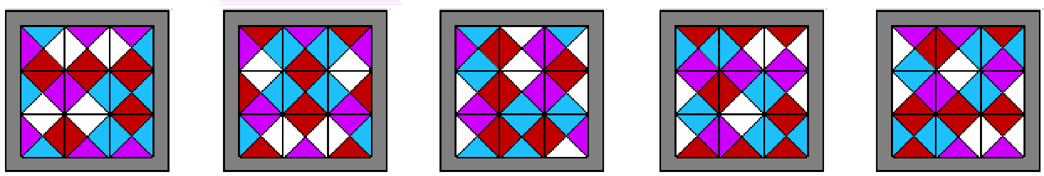


Figure 4: *Two colors enclosed simultaneously.*

Wrap-arounds. When opposite borders match at their row ends, the solution "wraps around", forming the equivalent of a **cylinder** (Figure 5a). When two opposite pairs of borders match, top to bottom and left to right, as in Figure 5b, the wrap-around can go both ways, forming the equivalent of a **torus**. When one pair of opposite borders is rotated instead of mirrored, “wrapping” them would require a half twist. Thus they represent a **Moebius strip** (Figure 5c). When one pair of opposite sides matches without twisting and the other pair is rotated as in Figure 5d, and all matching edges are joined, they represent a **Klein bottle**. When both pairs of opposite borders are rotated and joined by a half twist each, as in Figure 5e, they represent a **projective plane**.

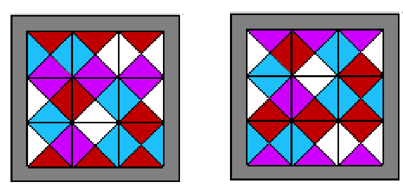


a) *b)* *c)* *d)* *e)*

Figure 5: *a) cylinder; b) torus; c) Moebius; d) Klein bottle; e) projective plane.*

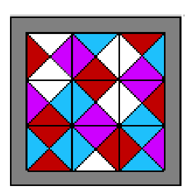
Wrap-around transformations. When rows are moved during a wrap-around sequence, the colors and numbers of squares enclosed can change. In Figure 6, moving the top row in the left square to the bottom yields the pattern on the right.

Figure 6: *Moving a row that wraps can change the enclosed landscape.*



No row ends match. It is curiously difficult to find a solution where none of the 6 rows has matched ends (Figure 7).

Figure 7: *No rows match at ends.*



Interior color divisions. The interior of a matched solution always shows 12 solid-color squares (diamonds). These interior color squares can contain 3 or 4 colors in all these sums, tabulated in Figure 8. The remaining two possibilities, 5 5 2 and 5 5 1 1, are not solvable because there are not enough tiles of the two minority colors (white and purple) to complete the border. For the special case of 4 4 2 2, any two colors can contribute the 4's, in all 6 possible pairings.

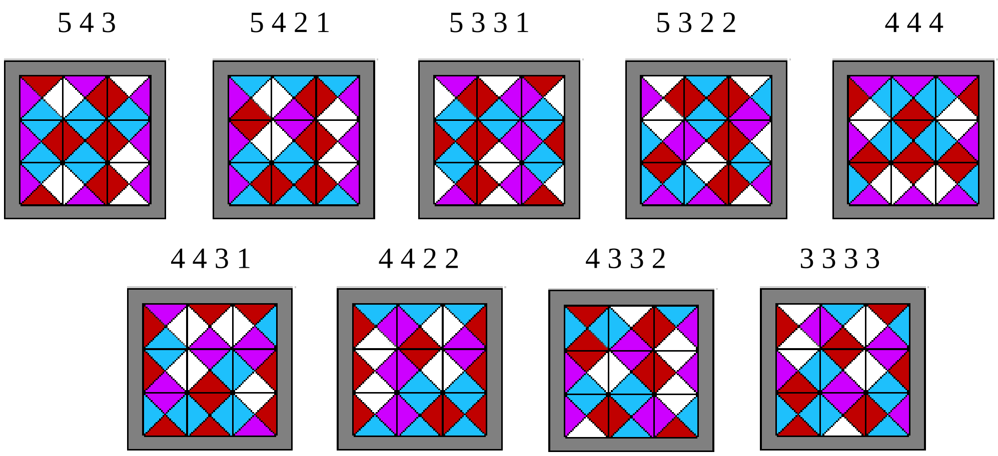


Figure 8: *Examples of all possible color frequencies on the enclosed 12 squares within the 3x3 square. Numbers refer to different colors based on frequency.*

Other observations.

1. Only one of the four colors can be entirely on the outside only (Figure 8, example 4 4 4).
2. The 9 squares can form 78 other symmetrical shapes with color-matching—the enneominoes that have vertical or rotational symmetry [4].
3. Several matched arrangements of the tiles can produce a perfect mirror symmetry of all colors (Figure 9). Note how the three bowtie pieces form the center column.

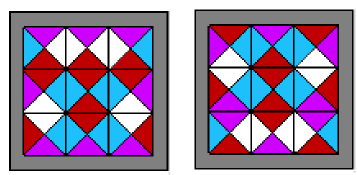


Figure 9: *Symmetrical matched arrangements of all four colors naturally produce double wrap-arounds as well.*

Non-matching arrangements. Exploring non-match solutions (samples shown in Figure 10) can be very liberating and can also produce color symmetries, an ever-changing work of art.

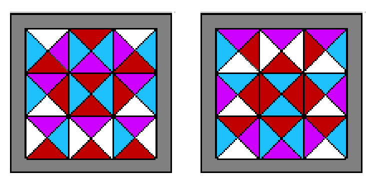


Figure 10: *Symmetrical non-matched arrangements have their own artistic charm.*

References

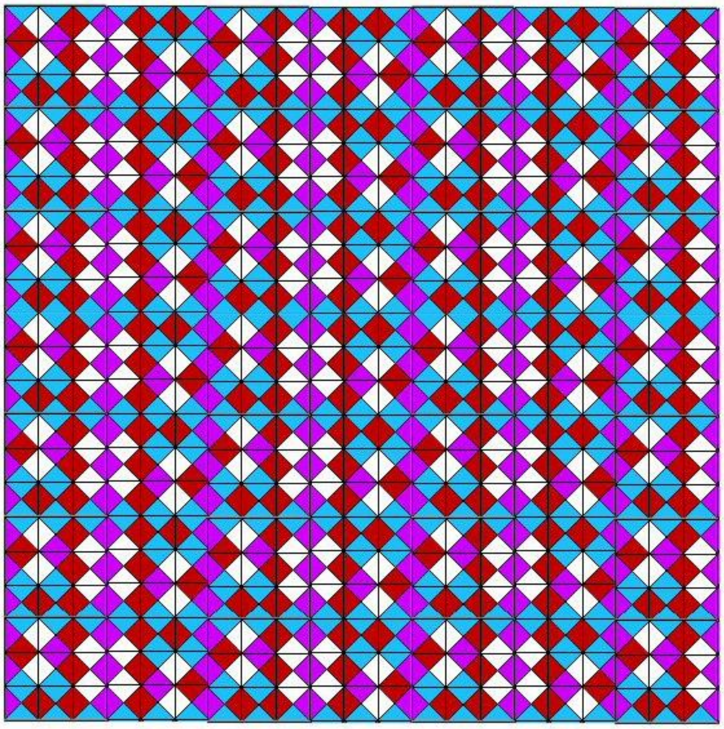
[1] Major Percy MacMahon. *New Mathematical Pastimes*, pp. 23-37. Cambridge Univ. Press, 1921. `https://archive.org/details/newmathematicalp00macmuoft` (as of April 15, 2017).

[2] Wade E. Philpott (1918-1985) Special Collections, University of Calgary. Of special interest is Philpott's pioneering work in computer-generated solution counts of 4x6 rectangles of MacMahon's Three-Color Squares. `http://www.ucalgary.ca/lib-old/SpecColl/philpott.htm` (as of April 15, 2017).

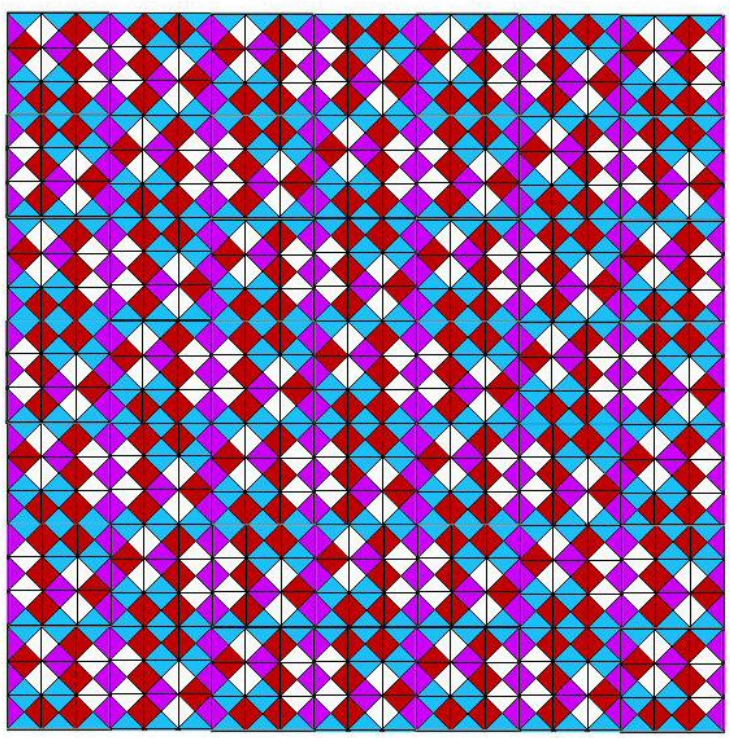
[3] Wade Philpott, Puzzle Explorer (biographical notes). `http://www.gamepuzzles.com/wade.htm` (as of April 15, 2017).

[4] *MiniMatch-I* (instruction manual). Kadon, 2015, `http://www.gamepuzzles.com/mim1-book.pdf` (as of April 15, 2017).

MiniMatch-I – mega-art from a single 3x3 array



Suggested by Joe Marasco
Designed by Kate Jones



Here are just two simple periodic assemblies. The total number of ways to arrange a 7x7 array of the single unit cell, which can have 8 possible orientations (4 rotations plus their reflections) is 8 to the 49th power (8^{49}). You do the math.

Since every possible matched solution of MiniMatch-I has not yet been identified, we cannot begin to guess how large is the repertoire of mega constructions. A small taste of the combinatorial possibilities of the Universe!